The original Collatz permutation of 1932: 3-adic dynamics and cycles

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Abstract

In a notebook entry dated 1 July 1932, Lothar Collatz considered a function defined modulo 3 that gives rise to a permutation of the positive integers, and asked whether the orbit containing the number 8 is finite or infinite. In this note we show that the function is a permutation, give its explicit inverse, prove that every orbit is either a finite cycle or a bi-infinite orbit tending to infinity in both directions, and verify the four known cycles. We also present the beginning of the orbit of 8 in both directions and some computational data. We prove that the extension of the function to the 3-adic integers is conjugate to the full shift on three symbols (the finite form of this fact is due to Matthews and Watts), and we obtain results on arbitrary cycles: an exact characterization of cycles by their sequence of residues, from which it follows that there are only finitely many cycles of each period; congruences for their minimum and maximum; bounds for the proportion of terms divisible by 3; an explicit bound for the minimum in terms of the distance from $L \log_2 3$ to the nearest integer; and a relation with the convergents of $\log_2 3$. The four known cycles correspond to four consecutive convergents; the result obtained accounts for this in the two shortest cases, but not in the other two. Combining the bounds with a computer search, we prove that the only cycles of period at most 5000 are the four known ones, and we explain why a verification of this kind cannot be extended as in the $3x + 1$ problem. The question about 8, and the existence of further cycles, remain open; this note does not attempt to settle them.

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1 Introduction

Before the $3n+1$ problem, Collatz studied other arithmetic functions defined by cases, representing them as directed graphs with an arrow from n to $f(n)$. As recorded by Lagarias [1], in his notebook of 1 July 1932 he considered the function

$$g(n) = \begin{cases} \dfrac{2n}{3}, & n \equiv 0 \pmod 3, \\ \dfrac{4n-1}{3}, & n \equiv 1 \pmod 3, \\ \dfrac{4n+1}{3}, & n \equiv 2 \pmod 3, \end{cases} \tag{1}$$

posed the problem of determining the cycle structure of the permutation it defines and asked, in particular, whether the orbit containing 8 is finite or infinite (in Lagarias's terminology, whether the "cycle" of the permutation containing 8 is finite or infinite). Ninety-four years later, the question remains unanswered.

The historical background, namely the relevant letters and the relation between this problem and the $3n+1$ problem, is discussed in [8]. Here we deal only with the function.

The function g is sequence A006369 of the OEIS [4]. Its inverse (Theorem 3.1) is the permutation that Conway called *amusical*, sequence A006368; Conway discussed the orbit of 8 in [5] and returned to it in [6], where he presents this question as a candidate for an arithmetical problem

that may be unsettleable. The problem was first published by Klamkin in 1963 [15], together with the question of whether g has only finitely many distinct orbits; Tavares [17] studies this last question through the density of the orbits. The problem also appears in Guy's collection [7, E17]. The permutation belongs to the class of residue-class-wise affine mappings, whose permutation groups are studied with algorithmic methods by Kohl [9].

The main results are in Section 8: the characterization of cycles by their word of residues (Theorem 8.8), based on the 3-adic conjugacy (Proposition 8.2); the constraints on the minimum, the maximum and the proportion of terms divisible by 3; the explicit bound for the minimum (Proposition 8.15); the relation with the convergents of $\log_2 3$ (Theorem 8.17); and the exclusion of cycles of period at most 5000 other than the known ones (Corollary 8.18). The preceding sections collect the elementary facts about the permutation, its known cycles and the orbit of 8.

Throughout, $\mathbb{N} = \{1, 2, 3, \dots\}$ and g^k denotes the k-th iterate of g, with $g^0 = \mathrm{id}$ and, once g is shown to be bijective, $g^{-k} = (g^{-1})^k$.

2 Integer form of the function

Lemma 2.1. *The function g takes values in $\mathbb{N}$ and, for every integer $k \geq 0$,*

$$g(3k) = 2k \quad (k \geq 1), \qquad g(3k+1) = 4k+1, \qquad g(3k+2) = 4k+3.$$

Proof. Substitute into (1): $g(3k) = 2 \cdot 3k/3 = 2k$; $g(3k+1) = (12k+4-1)/3 = 4k+1$; $g(3k+2) = (12k+8+1)/3 = 4k+3$. All three values are positive integers. □

Remark 2.2. In particular, iterating g never produces fractions: the numerators $4n \mp 1$ are always divisible by 3 in the corresponding branch. The form of Lemma 2.1 is the one used by Lagarias [1]; it is not a modification of the original function, but the same function written without denominators.

3 The function is a permutation

Theorem 3.1. *The function $g\colon \mathbb{N} \to \mathbb{N}$ is bijective, and its inverse is*

$$g^{-1}(m) = \begin{cases} \dfrac{3m}{2}, & m \equiv 0 \pmod 2, \\ \dfrac{3m+1}{4}, & m \equiv 1 \pmod 4, \\ \dfrac{3m-1}{4}, & m \equiv 3 \pmod 4. \end{cases}$$

Proof. By Lemma 2.1, g maps:

- the class $\{3k : k \geq 1\}$ bijectively onto the even numbers $\{2k : k \geq 1\}$;
- the class $\{3k+1 : k \geq 0\}$ bijectively onto $\{4k+1 : k \geq 0\}$;
- the class $\{3k+2 : k \geq 0\}$ bijectively onto $\{4k+3 : k \geq 0\}$.

The three source classes form a partition of $\mathbb{N}$, and so do the three target classes (even numbers, odd numbers $\equiv 1$ and odd numbers $\equiv 3 \pmod 4$). Since each piece is mapped bijectively onto its image, g is bijective. The inverse is obtained by solving for k in each piece: if $m = 2k$, then $g^{-1}(m) = 3k = 3m/2$; if $m = 4k+1$, then $g^{-1}(m) = 3k+1 = (3m+1)/4$; if $m = 4k+3$, then $g^{-1}(m) = 3k+2 = (3m-1)/4$. □

Remark 3.2. Writing $m = 2k$, $m = 4k+1$ and $m = 4k-1$, the inverse reads $g^{-1}(2k) = 3k$, $g^{-1}(4k+1) = 3k+1$, $g^{-1}(4k-1) = 3k-1$: it is Conway's amusical permutation [5, 6], sequence A006368 of the OEIS, restricted to $\mathbb{N}$.

Observation 3.3. The class of n modulo 3 determines the parity of $g(n)$, and the class of m modulo 4 determines the branch of g^{-1}. Going forward, g multiplies approximately by $2/3$ or by $4/3$; going backward, g^{-1} multiplies approximately by $3/2$ or by $3/4$.

4 Dichotomy of orbits

Proposition 4.1. *Let $n \in \mathbb{N}$. Exactly one of the following holds:*

(a) there exists $p \geq 1$ with $g^p(n) = n$, and the orbit of n is a finite cycle;

(b) the numbers $g^k(n)$, $k \in \mathbb{Z}$, are pairwise distinct, and $g^k(n) \to \infty$ both as $k \to +\infty$ and as $k \to -\infty$.

Proof. If $g^a(n) = g^b(n)$ with $a < b$, applying g^{-a} (Theorem 3.1) gives $g^{b-a}(n) = n$, which is case (a). Otherwise the $g^k(n)$ are pairwise distinct. Given a bound B, there are only B positive integers $\leq B$, and each appears at most once in the sequence; hence $g^k(n) > B$ except for finitely many values of k. This gives (b). □

Corollary 4.2. *For the orbit of 8, the alternatives posed in 1932 (that the orbit "becomes cyclic" or that it "diverges") are exactly the two cases of Proposition 4.1: either 8 belongs to a finite cycle, or $g^k(8) \to \infty$ in both directions. There is no third possibility: two distinct orbits are disjoint, so the orbit of 8 cannot fall into one of the known cycles.*

5 The known cycles

Proposition 5.1. *The function g has the following four cycles:*

$$(1), \qquad (2,\ 3), \qquad (4,\ 5,\ 7,\ 9,\ 6), \qquad (44,\ 59,\ 79,\ 105,\ 70,\ 93,\ 62,\ 83,\ 111,\ 74,\ 99,\ 66).$$

Proof. Direct verification with Lemma 2.1. For the long cycle:

n	$n \bmod 3$	rule	$g(n)$
44	2	$4k+3$, $k=14$	59
59	2	$4k+3$, $k=19$	79
79	1	$4k+1$, $k=26$	105
105	0	$2k$, $k=35$	70
70	1	$4k+1$, $k=23$	93
93	0	$2k$, $k=31$	62
62	2	$4k+3$, $k=20$	83
83	2	$4k+3$, $k=27$	111
111	0	$2k$, $k=37$	74
74	2	$4k+3$, $k=24$	99
99	0	$2k$, $k=33$	66
66	0	$2k$, $k=22$	44

The three short cycles are checked in the same way: $g(1)=1$; $g(2)=3$, $g(3)=2$; $g(4)=5$, $g(5)=7$, $g(7)=9$, $g(9)=6$, $g(6)=4$. The cycles of lengths 5 and 12 are sequences A094328 and A185590 of the OEIS. □

Remark 5.2. The short cycles contain $1, 2, \ldots, 7$ and 9, so 8 is the smallest positive integer not belonging to any of the known cycles. This is why it is the first non-trivial case of the question.

6 The orbit of 8

Going forward, the first terms are (sequence A028394 of the OEIS)

8, 11, 15, 10, 13, 17, 23, 31, 41, 55, 73, 97, 129, 86, 115, 153, 102, 68, 91, 121, 161, 215, 287, 383, 511, 681, 454, 605, 807, 538, 717, 478, 637, 849, 566, 755, 1007, 1343, 1791, 1194, 796, 1061, ...

and going backward, applying g^{-1} (sequence A028393),

8, 12, 18, 27, 20, 30, 45, 34, 51, 38, 57, 43, 32, 48, 72, 108, ...

The question about 8 is equivalent to asking whether the forward and backward orbits meet again away from the initial term 8. Indeed, since g is bijective, an equality

$$g^r(8) = g^{-s}(8), \qquad r, s > 0,$$

is equivalent to $g^{r+s}(8) = 8$.

Observation 6.1 (computational data). We computed 10^6 iterations in each direction starting from 8. The number 8 does not reappear in either direction. After 10^6 steps, $g^{10^6}(8)$ has 24,282 decimal digits and $g^{-10^6}(8)$ has 25,655. Along the forward orbit, the three classes modulo 3 occur with frequencies 0.3344, 0.3327 and 0.3329.

Remark 6.2 (heuristic, not a proof). The three branches of g multiply asymptotically by 2/3, 4/3 and 4/3 (the branches $4k+1$ and $4k+3$ do not multiply exactly by 4/3). If the classes modulo 3 of the terms behaved as independent and equiprobable, the mean asymptotic multiplicative factor per forward step would be

$$\left(\tfrac{2}{3} \cdot \tfrac{4}{3} \cdot \tfrac{4}{3}\right)^{1/3} = \left(\tfrac{32}{27}\right)^{1/3} > 1,$$

which predicts about $10^6 \cdot \frac{1}{3}\log_{10}\frac{32}{27} \approx 24{,}595$ digits after 10^6 steps. Going backward, with equiprobable classes modulo 4, the mean factor would be $(3/2)^{1/2}(3/4)^{1/2} = (9/8)^{1/2}$, which predicts about 25,576 digits. Both predictions agree with the values of Observation 6.1. This suggests that the orbit of 8 is infinite, but does not prove it: nothing guarantees that the particular orbit of 8 behaves like the random model. Observation 8.7 shows that this model is exactly the statistics of g extended to $\mathbb{Z}_3$ with its natural measure.

7 Search for further cycles

Proposition 7.1 (computational verification)**.** *Every cycle of g of length at most* 5000 *whose minimum is at most* 10^5 *is one of the four of Proposition 5.1.*

Method. If a cycle has minimum m and length ℓ, the orbit of m stays strictly above m during steps $1, \dots, \ell - 1$ and returns to m at step ℓ. For each $m \le 10^5$ we iterate g from m until the value returns to m, drops below m, or 5000 steps are reached. The orbit returns to m only for $m = 1, 2, 4, 44$, with lengths $1, 2, 5, 12$. The code is given in Appendix A. □

Remark 7.2. Proposition 7.1 does not by itself exclude cycles all of whose elements are large; for periods at most 5000 this is done by Corollary 8.18. Among the integers not lying in the known cycles, the smallest after 8 whose orbit has not been seen to join that of 8 is 14: it does not appear in the first $2 \cdot 10^4$ steps of the orbit of 8 in either direction. Its orbit begins $14, 19, 25, 33, 22, 29, 39, 26, 35, 47, 63, 42, 28, \dots$ (sequence A028396) and poses, in principle, a separate question. Simons [11] gives, from his computations, $8, 14, 40, 64, 80, 82, \dots$ as minima of distinct divergent trajectories. In the formulation of the problem proposed by Lagarias in [2], part (b) records the conjecture that the bi-infinite orbit of 8 has density zero in $\mathbb{N}$, that is,

$$\lim_{N\to\infty} \frac{|S_N|}{N} = 0, \qquad S_N = \{n \in [1, N] : n = g^k(8) \text{ for some } k \in \mathbb{Z}\},$$

and part (c) states as an open problem to prove that there are infinitely many positive integers outside that orbit. Tavares [17] proves that an orbit has density zero if and only if, for some x in the orbit and some N, the set of terms $g^{jN}(x)$, $j \in \mathbb{Z}$, at which g^N decreases has density zero; he also records the indication of Venturini [18, p. 303] that one of his identities might be used to prove that every orbit has density zero. If every orbit had density zero, g would have infinitely many distinct orbits. For our part, the computation above does not exclude that 14 belongs to the orbit of 8: it only shows that 14 does not appear in its first $2 \cdot 10^4$ steps in each direction.

8 Arithmetic constraints on cycles

In this section we obtain necessary conditions that any cycle of g must satisfy. We write Lemma 2.1 in the form

$$3\,g(n) = c(n)\,n + \varepsilon(n), \qquad (c(n), \varepsilon(n)) = \begin{cases} (2, 0), & n \equiv 0 \pmod 3, \\ (4, -1), & n \equiv 1 \pmod 3, \\ (4, +1), & n \equiv 2 \pmod 3. \end{cases} \tag{2}$$

8.1 The equation of a cycle

Proposition 8.1. *Let $(n_0, n_1, \dots, n_{L-1})$ be a cycle of g, with $g(n_i) = n_{i+1}$ and indices modulo L. Put $c_i = c(n_i)$, $\varepsilon_i = \varepsilon(n_i)$, let a be the number of terms of the cycle divisible by 3, and let $K = 2L - a$. Then*

$$\left(3^L - 2^K\right) n_0 = \sum_{i=0}^{L-1} \varepsilon_i \, 3^i \prod_{j=i+1}^{L-1} c_j. \tag{3}$$

Proof. By induction on k, from (2) one obtains

$$3^k n_k = \Big(\prod_{i<k} c_i\Big) n_0 + \sum_{i<k} \varepsilon_i \, 3^i \prod_{i<j<k} c_j,$$

since $3^{k+1} n_{k+1} = c_k \, (3^k n_k) + \varepsilon_k \, 3^k$. For $k = L$ we have $n_L = n_0$, and $\prod_{i<L} c_i = 2^a 4^{L-a} = 2^K$. □

The formula for $3^k n_k$ in the proof is the one given by Matthews and Watts for generalized Collatz mappings [16, Lemma 1], which Tavares applies to this permutation [17, Lemma 8].

For the four known cycles, taking n_0 to be the minimum, (3) gives $(-1) \cdot 1 = -1$, $1 \cdot 2 = 2$, $(-13) \cdot 4 = -52$ and $7153 \cdot 44 = 314{,}732$.

8.2 The 3-adic dynamics

The formulas of Lemma 2.1 define g on the ring $\mathbb{Z}_3$ of 3-adic integers, and (2) remains valid there. Let σ be the shift on sequences of three symbols, $\sigma(r_0, r_1, r_2, \dots) = (r_1, r_2, \dots)$, and let

$$\mathcal{I} \colon \mathbb{Z}_3 \longrightarrow \{0,1,2\}^{\mathbb{N}_0}, \qquad \mathcal{I}(x) = (x \bmod 3,\ g(x) \bmod 3,\ g^2(x) \bmod 3,\ \dots)$$

be the map assigning to each x its sequence of residues.

Proposition 8.2. *The map $\mathcal{I}$ is a homeomorphism and satisfies $\mathcal{I} \circ g = \sigma \circ \mathcal{I}$. That is, g acting on $\mathbb{Z}_3$ is conjugate to the full shift on three symbols.*

Proof. If $x \equiv i \pmod 3$, then $g(x + 3^t u) = g(x) + c \, 3^{t-1} u$ with $c \in \{2, 4\}$, which is a unit modulo 3. From this one obtains, by induction on j, that the map

$$x \bmod 3^j \ \longmapsto\ (x \bmod 3,\ g(x) \bmod 3,\ \dots,\ g^{j-1}(x) \bmod 3)$$

is well defined and is a bijection from $\mathbb{Z}/3^j\mathbb{Z}$ onto $\{0,1,2\}^j$: if two classes give the same sequence, they have the same residue i, can be written $x = i + 3t$, $x' = i + 3t'$, and $g(x) - g(x') = c\,(t - t')$ with $g(x) \equiv g(x') \pmod{3^{j-1}}$ by induction, hence $t \equiv t' \pmod{3^{j-1}}$. These bijections are compatible as j varies, so $\mathcal{I}$ is bijective. It is continuous, since its first j coordinates depend only on $x \bmod 3^j$, and a continuous bijection from a compact space onto a Hausdorff space is a homeomorphism. The identity $\mathcal{I} \circ g = \sigma \circ \mathcal{I}$ is immediate from the definition. □

Remark 8.3. The bijection modulo 3^j used in the proof is, in a more general setting, Corollary 2.2 of Matthews and Watts [16]; Tavares proves it for this permutation [17, Lemma 9]. Proposition 8.2 expresses that fact in the language of symbolic dynamics.

Remark 8.4. On $\mathbb{Z}_3$, g is three-to-one, like the shift: each of the three branches is surjective, because 2 and 4 are units of $\mathbb{Z}_3$. That g is a permutation of $\mathbb{N}$ (Theorem 3.1) depends on how the images are distributed modulo 4, a 2-adic phenomenon invisible in $\mathbb{Z}_3$. Proposition 8.2 is the analogue, for this permutation, of the conjugacy of the $3x+1$ function on $\mathbb{Z}_2$ with the shift on two symbols [1]. Weitzer [10] has proposed a general framework, p-adic systems, inspired by the $3n+1$ problem, within which descriptions of this kind can be placed.

Corollary 8.5. *For each $L \geq 1$, the equation $g^L(x) = x$ has exactly 3^L solutions in $\mathbb{Z}_3$, one for each word of length L. The number of points of exact period L is $P_L = \sum_{d|L} \mu(L/d)\, 3^d$, and the number of orbits of exact period L is P_L/L.*

Proof. By Proposition 8.2, the fixed points of g^L correspond to those of σ^L, which are the periodic sequences with period dividing L, determined by their first L symbols. The formula for P_L follows by Möbius inversion from $\sum_{d|L} P_d = 3^L$. □

Observation 8.6. The periodic points of g are dense in $\mathbb{Z}_3$, since under the conjugacy of Proposition 8.2 they correspond to the periodic sequences, which are dense in the full shift. The known scarcity of cycles in $\mathbb{N}$ therefore does not come from a scarcity of periodic orbits in the 3-adic extension, but from the additional arithmetic condition that the periodic point belong to $\mathbb{N}$.

Observation 8.7. Let μ be the normalized Haar measure on $\mathbb{Z}_3$. For each word $(r_0, \ldots, r_{j-1})$, the set of x whose first j residues are $r_0, \ldots, r_{j-1}$ is a single class modulo 3^j, and therefore has measure 3^{-j}. Thus, under μ, the residues $x \bmod 3$, $g(x) \bmod 3$, $g^2(x) \bmod 3, \ldots$ are independent random variables, uniformly distributed on $\{0,1,2\}$. The random model of Remark 6.2 is therefore not an arbitrary choice: it is exactly the statistics of g extended to $\mathbb{Z}_3$ with its natural measure. This says nothing about the orbit of a particular integer, such as 8: the positive integers form a set of measure zero in $\mathbb{Z}_3$.

8.3 Characterization of integer cycles

Identity (3) uses only the sequence of residues of the first L terms, and its proof holds unchanged in $\mathbb{Z}_3$. For a word $w = (r_0, \ldots, r_{L-1}) \in \{0,1,2\}^L$ we define c_i, ε_i by (2) with r_i in place of $n \bmod 3$, $a = \#\{i : r_i = 0\}$, $K = 2L - a$, and

$$C(w) = \sum_{i=0}^{L-1} \varepsilon_i\, 3^i \prod_{j=i+1}^{L-1} c_j, \qquad n(w) = \frac{C(w)}{3^L - 2^K}.$$

Since $a \leq L$, we have $K \geq L \geq 1$, so $3^L - 2^K$ is odd minus even and does not vanish; moreover, it is not divisible by 3.

Theorem 8.8. *Let $w \in \{0,1,2\}^L$. There exists a positive integer n_0 with $g^L(n_0) = n_0$ whose residues $n_0, g(n_0), \ldots, g^{L-1}(n_0)$ modulo 3 form the word w if and only if $n(w)$ is a positive integer, and in that case $n_0 = n(w)$.*

Proof. By Proposition 8.2, there is a unique $x_w \in \mathbb{Z}_3$ with $\mathcal{I}(x_w) = www\cdots$, and it satisfies $g^L(x_w) = x_w$ because σ^L fixes that sequence. Identity (3) in $\mathbb{Z}_3$ gives $(3^L - 2^K)\, x_w = C(w)$, and since $3^L - 2^K$ is a unit of $\mathbb{Z}_3$, we conclude that $x_w = n(w)$.

If $n(w)$ is a positive integer, then $n_0 = n(w) = x_w$ satisfies $g^L(n_0) = n_0$ and has the word w. Conversely, if a positive integer n_0 satisfies $g^L(n_0) = n_0$ with word w, its sequence of residues is $www\cdots$, so $n_0 = x_w = n(w)$. □

There is therefore no need to check afterwards that the residues of $n(w)$ reproduce the word w: it suffices that $n(w)$ be a positive integer. In other words, in $\mathbb{Z}_3$ all periodic itineraries exist (Corollary 8.5); the problem of the cycles of g is to decide which of these 3-adic periodic points are positive integers.

Theorem 8.8 is the form taken for this permutation by a fact known for the $3x+1$ problem: each periodic parity sequence determines a unique rational periodic point, and there is an integer cycle with that sequence if and only if this rational number is an integer (Böhm and Sontacchi [13]; Lagarias [14]). The composition formula on which it rests is that of Matthews and Watts [16].

Observation 8.9. Let $\rho w = (r_1, \dots, r_{L-1}, r_0)$ be the rotation of w, and let c_0, ε_0 be the values corresponding to r_0. Then

$$3\,C(\rho w) = c_0\,C(w) + \varepsilon_0\,(3^L - 2^K).$$

Indeed, ρw has the same number of zeros as w, so the denominator of $n(\rho w)$ is the same; and since $\mathcal{I}(g(x_w)) = \sigma(www\cdots) = (\rho w)(\rho w)\cdots$, we have $n(\rho w) = g(n(w)) = (c_0\,n(w) + \varepsilon_0)/3$ by (2), which is equivalent to the stated identity. Since $3^L - 2^K$ is coprime to 3, it follows that $3^L - 2^K$ divides $C(w)$ if and only if it divides $C(\rho w)$: the integrality condition of Theorem 8.8 does not depend on which point of the cycle is taken as the starting point, and rotation of the word acts on the quotients exactly as g.

Corollary 8.10. *For each $L \geq 1$ there are at most 3^L positive integers n with $g^L(n) = n$. In particular, there are only finitely many cycles whose period divides L, and at most $3^L/L$ cycles of exact period L.*

Proof. Every positive integer with $g^L(n) = n$ is one of the 3^L 3-adic solutions of Corollary 8.5. Each cycle of exact period L contributes L of these integers. □

Observation 8.11. The formulas of Lemma 2.1 satisfy $g(-n) = -g(n)$ for all $n \in \mathbb{Z}$: the change of sign exchanges the classes 1 and 2 modulo 3 and fixes the class 0. In terms of words, if $\bar{w}$ is obtained from w by exchanging the symbols 1 and 2, the c_i do not change, the ε_i change sign, and therefore $C(\bar{w}) = -C(w)$ and $n(\bar{w}) = -n(w)$. Each cycle of positive integers thus has a reflected cycle of negative integers. Together with Observation 8.9, the two natural operations on words act on periodic points as $n(\rho w) = g(n(w))$ and $n(\bar{w}) = -n(w)$.

8.4 The minimum and the maximum of a cycle

Proposition 8.12. *Let a cycle of g have length $L \geq 2$, minimum m and maximum M. Then*

$$m \equiv 2 \text{ or } 4 \pmod 6, \qquad M \equiv 3 \pmod 6.$$

Proof. For $n \geq 2$ we have $g(n) < n$ if and only if $n \equiv 0 \pmod 3$, since $(4n-1)/3 > n$ for $n > 1$ and $(4n+1)/3 > n$ always. Since 1 is a fixed point, no term of a cycle of length $L \geq 2$ equals 1.

The predecessor p of m in the cycle is different from m and satisfies $g(p) = m < p$; hence $p = 3k$ and $m = 2k$ is even. Moreover $g(m) > m$, so $m \not\equiv 0 \pmod 3$. Therefore $m \equiv 2$ or $4 \pmod 6$.

Since $g(M) < M$, we have $M \equiv 0 \pmod 3$. Its predecessor q satisfies $g(q) = M > q$, hence $q \not\equiv 0 \pmod 3$ and $M = g(q)$ is of the form $4k+1$ or $4k+3$, that is, odd. Therefore $M \equiv 3 \pmod 6$. □

The known cycles satisfy this: their minima $2, 4, 44$ are $\equiv 2, 4, 2 \pmod 6$, and their maxima $3, 9, 111$ are $\equiv 3 \pmod 6$.

8.5 The proportion of the contracting branch

Proposition 8.13. *Let a cycle of g have length L and minimum m, and let a be the number of its terms divisible by 3. With*

$$A_\pm = \frac{4m \pm 1}{3m}, \qquad \Phi(A) = \frac{\ln A}{\ln A - \ln(2/3)},$$

we have

$$\Phi(A_-) \le \frac{a}{L} \le \Phi(A_+). \tag{4}$$

As $m \to \infty$, both bounds tend to $\dfrac{\ln(4/3)}{\ln 2} = 2 - \log_2 3 = 0.41503\ldots$

Proof. Around the cycle we have the exact identity $\prod_{i=0}^{L-1} g(n_i)/n_i = \prod_i n_{i+1}/n_i = 1$. For $n \ge m$, the quotient $g(n)/n$ equals $2/3$ if $3 \mid n$, and lies between A_- and A_+ otherwise, since it equals $4/3 \mp 1/(3n)$. Therefore

$$\left(\tfrac{2}{3}\right)^a A_-^{L-a} \le 1 \le \left(\tfrac{2}{3}\right)^a A_+^{L-a}. \tag{5}$$

Taking logarithms in the first inequality, $L \ln A_- \le a(\ln A_- - \ln \frac{2}{3})$, and since $A_- > 2/3$ we obtain $a/L \ge \Phi(A_-)$. The second inequality similarly gives $a/L \le \Phi(A_+)$. The limit follows from $A_\pm \to 4/3$. □

Remark 8.14. For the known cycles, a/L equals 0, $1/2$, $2/5$ and $5/12 = 0.4166\ldots$; for the last one, (4) with $m = 44$ gives the interval $[0.4102, 0.4198]$. By contrast, along the orbit of 8 the class 0 occurs with frequency 0.3344 (Observation 6.1), close to $1/3$. For a cycle with a large minimum to close, the contracting branch must be over-represented, with a frequency close to 0.415, which the orbit of 8 does not show in the computed steps. This proves nothing about the orbit of 8, but separates the observed regime from the regime required for closure.

8.6 An explicit bound for the minimum

We write $\|\xi\|$ for the distance from a real number ξ to the nearest integer.

Proposition 8.15. *Let a cycle of g have period L and minimum m, with a terms divisible by 3, and let $K = 2L - a$, $q = L - a$ and $\Delta = K \ln 2 - L \ln 3$. Then $q \ge 1$, $\Delta \ne 0$ and*

$$m \le M(L, a) = \begin{cases} \dfrac{1}{4\left(1 - e^{-\Delta/q}\right)}, & \Delta > 0, \\[2ex] \dfrac{1}{4\left(e^{|\Delta|/q} - 1\right)}, & \Delta < 0. \end{cases} \tag{6}$$

Moreover $M(L, a) \le \dfrac{q}{4|\Delta|} + \dfrac{1}{4}$, and consequently

$$m \le \frac{L}{4 \ln 2 \, \|L \log_2 3\|} + \frac{1}{4}. \tag{7}$$

Proof. If all terms of the cycle were divisible by 3, each step would multiply by $2/3$ and the cycle could not close; hence $q \geq 1$. Since $2^K \neq 3^L$, we have $\Delta \neq 0$. With $x = 1/(4m)$ we have $A_\pm = \frac{4}{3}(1 \pm x)$ and $(2/3)^a(4/3)^q = 2^K/3^L = e^\Delta$, so (5) reads

$$(1+x)^{-q} \leq e^\Delta \leq (1-x)^{-q}.$$

If $\Delta > 0$, the right-hand inequality gives $1 - x \leq e^{-\Delta/q}$, that is, $x \geq 1 - e^{-\Delta/q}$; if $\Delta < 0$, the left-hand one gives $1 + x \geq e^{|\Delta|/q}$, that is, $x \geq e^{|\Delta|/q} - 1$. This is (6).

With $t = |\Delta|/q$ we have $1 - e^{-t} \geq t/(1+t)$ and $e^t - 1 \geq t$, both equivalent to $e^t \geq 1 + t$; in both cases $M(L, a) \leq (1+t)/(4t) = q/(4|\Delta|) + 1/4$. Finally, $|\Delta| = \ln 2\, |K - L \log_2 3| \geq \ln 2\, \|L \log_2 3\|$ and $q \leq L$. □

Remark 8.16. The bound is attained by the cycle $(2, 3)$: with $(L, a) = (2, 1)$ we have $\Delta = \ln(8/9)$ and $M(2, 1) = 2 = m$. For the cycle of length 12, $M(12, 5) \approx 129.0$, compared with $m = 44$. For $(L, a) = (41, 17)$, which corresponds to the convergent $65/41$, we obtain $M(41, 17) \approx 523.6$: a cycle of period 41 would have minimum at most 523, and Proposition 7.1 excludes it. The convergent $65/41$ approximates $\log_2 3$ well, but not well enough to allow a cycle with a large minimum.

Bound (7) makes visible the role of Diophantine approximation: a cycle of period L with a large minimum requires $L \log_2 3$ to be very close to an integer, with $\|L \log_2 3\|$ at most of the order of L/m. The convergents and intermediate convergents of $\log_2 3$, which provide particularly good rational approximations, thus appear naturally. Atkin (1966) obtained for this permutation a bound of this kind for the minimum of a cycle, in terms of the period and the number of terms not divisible by 3; Tremblay [12] reproduces his proof and tabulates the periods for which that bound is largest, among them 41, 53, 94, 306, 665, 971, 1636 and 14,936. Analogous bounds are known for the $3x + 1$ problem and its variants [3].

This bound complements the 3-adic characterization: Corollary 8.10 bounds the number of cycles of each period, and (7) bounds the region where they can lie, since $\|L \log_2 3\| > 0$ by the irrationality of $\log_2 3$.

8.7 Cycles and convergents of $\log_2 3$

Theorem 8.17. *Let a cycle of g have length L and minimum m, and let $K = 2L - a$ as in Proposition 8.1. Then*

$$\left|\log_2 3 - \frac{K}{L}\right| \leq \frac{L-a}{L}\left(-\log_2\left(1 - \frac{1}{4m}\right)\right) \leq \frac{L-a}{L} \cdot \frac{1}{(4m-1)\ln 2} \leq \frac{1}{(4m-1)\ln 2}. \tag{8}$$

If

$$-\ln\left(1 - \frac{1}{4m}\right) < \frac{\ln 2}{2L(L-a)}, \tag{9}$$

then the fraction K/L, once reduced, is a convergent of the continued fraction expansion of $\log_2 3$. This holds in particular if $L \leq \sqrt{m}$.

Proof. Let $x = 1/(4m)$, so that $A_\pm = \frac{4}{3}(1 \pm x)$. Since $(2/3)^a(4/3)^{L-a} = 2^K/3^L$, inequalities (5) read

$$(1+x)^{-(L-a)} \leq \frac{2^K}{3^L} \leq (1-x)^{-(L-a)}.$$

Taking base-2 logarithms and using $\log_2(1+x) \le -\log_2(1-x)$,

$$\left|K - L\log_2 3\right| \le (L-a)(-\log_2(1-x)) \le (L-a)\,\frac{x}{(1-x)\ln 2},$$

where the last inequality uses $-\ln(1-x) \le x/(1-x)$. Dividing by L gives (8), since $x/(1-x) = 1/(4m-1)$ and $L - a \le L$.

Condition (9) is equivalent to the middle term of (8) being less than $1/(2L^2)$. If $L \le \sqrt{m}$, then $1/(2L^2) \ge 1/(2m)$, and for every $m \ge 1$ we have $1/((4m-1)\ln 2) < 1/(2m)$, since this is equivalent to $(2\ln 2 - 1)\cdot 2m > \ln 2$, which holds for $m \ge 1$. In both cases $|\log_2 3 - K/L| < 1/(2L^2)$. Write $K/L = p/q$ in lowest terms, with $q \le L$. Then

$$\left|\log_2 3 - \frac{p}{q}\right| < \frac{1}{2L^2} \le \frac{1}{2q^2},$$

and by Legendre's theorem, according to which every reduced fraction p/q with $|\xi - p/q| < 1/(2q^2)$, for irrational ξ, is a convergent of ξ, the fraction p/q is a convergent of $\log_2 3$. □

Corollary 8.18 (computer-assisted proof)**.** *The only cycles of g of period at most* 5000 *are the four of Proposition 5.1.*

Proof. Let a cycle have period $L \le 5000$, minimum m, and a terms divisible by 3. If $m \le 10^5$, Proposition 7.1 suffices. Suppose $m > 10^5$. Multiplying (5) by $3^L m^{L-a}$, we obtain the integer inequalities

$$2^a(4m-1)^{L-a} \;\le\; 3^L m^{L-a} \;\le\; 2^a(4m+1)^{L-a}, \tag{10}$$

which can be checked with exact arithmetic, without logarithms or rounding. For fixed L and a, the ratio of the left-hand side to the middle term is $(2/3)^a A_-^{L-a}$, which increases with m, and the ratio of the right-hand side to the middle term is $(2/3)^a A_+^{L-a}$, which decreases with m; so if (10) fails for $m = 100{,}001$, it fails for all $m \ge 100{,}001$. Moreover, both ratios decrease with a, because $2/3 < A_- < A_+$, so the values of a satisfying (10) form an interval of integers, which can be located by bisection. The exact check for $1 \le L \le 1000$ and $m = 100{,}001$ leaves only $(L,a) = (665, 276)$ and $(L,a) = (971, 403)$. By the same monotonicity, (10) holds for $(665, 276)$ exactly up to $m = 2{,}227{,}739$ and for $(971, 403)$ exactly up to $m = 145{,}107$. These limits are the integer parts of $M(665, 276) \approx 2{,}227{,}739.68$ and $M(971, 403) \approx 145{,}107.60$ in Proposition 8.15; already the uniform bound (7) exceeds 10^5, for $L \le 1000$, only when $L = 665$ or $L = 971$. The exact check by means of (10) is what the proof rests on. It remains to check the minima $100{,}001 \le m \le 2{,}227{,}739$ with $L = 665$ and $100{,}001 \le m \le 145{,}107$ with $L = 971$, which by Proposition 8.12 satisfy $m \equiv 2$ or $4 \pmod 6$. For each of them, the orbit of m drops below m or does not return to m within L steps, so none of them is the minimum of a cycle. The same procedure for $1001 \le L \le 5000$ leaves 51 pairs (L,a), whose exact bounds for the minimum range from 100,593 to 2,227,739; the maximum is attained by the multiples of 665 with $K/L = 1054/665$. For each pair we run through the minima $100{,}001 \le m \le$ bound with $m \equiv 2$ or $4 \pmod 6$, stopping the iteration as soon as the orbit drops below m, L steps are reached, more than a terms divisible by 3 occur, or more than $L - a$ terms not divisible by 3 occur; the last two conditions cannot happen before closure in a cycle with parameters (L,a). No m turns out to be the minimum of a cycle. The search function of the appendix accepts any return to m, without requiring it to occur exactly at step L with exactly a terms divisible by 3; a return with other parameters would also be a new cycle with minimum greater than 10^5, so this check is stricter than necessary and cannot miss any cycle. The code is given in Appendix A. The search was repeated with an independent program, which obtains the pairs (L,a) from the bound (6) confirmed with (10) and discards candidates by a size bound (even with all the remaining contracting steps, the value could not return to m) instead of by composition; it gave the same 53 pairs and no return to m. □

Remark 8.19. For the survivor $(665, 276)$ we have $K/L = 1054/665$, a convergent of $\log_2 3$; for the other, $K/L = 1539/971$, which is not a convergent but an intermediate convergent, the mediant of $485/306$ and $1054/665$. The sieve of Proposition 8.13 is sharper here than the Legendre condition of Theorem 8.17. Between 1001 and 5000 the pair $(L, a) = (1024, 425)$ also survives, with $K/L = 1623/1024$, which is neither a convergent nor an intermediate convergent: the band left by the sieve admits fractions of other kinds. The convergents of $\log_2 3 = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, \dots]$ are

$$\tfrac{1}{1},\ \tfrac{2}{1},\ \tfrac{3}{2},\ \tfrac{8}{5},\ \tfrac{19}{12},\ \tfrac{65}{41},\ \tfrac{84}{53},\ \tfrac{485}{306},\ \tfrac{1054}{665},\ \tfrac{24727}{15601},\ \dots$$

Observation 8.20. The four known cycles correspond to four consecutive convergents of $\log_2 3$:

cycle	L	a	K/L
(1)	1	0	2/1
$(2, 3)$	2	1	3/2
$(4, 5, 7, 9, 6)$	5	2	8/5
length 12 (minimum 44)	12	5	19/12

The fixed point and the cycle $(2, 3)$ satisfy condition (9) of Theorem 8.17, so for them the correspondence is forced; the cycles of lengths 5 and 12 do not satisfy it, and for them the coincidence is an observed fact and not a consequence of the theorem.

The relation between the lengths $1, 2, 5, 12$ and the continued fraction expansion of $\log_2 3$ was already observed by Shanks in 1965, as recorded in Lagarias's annotated bibliography [3]. Atkin (1966) qualified this heuristic with an example: the related permutation $f(3n) = 4n + 3$, $f(3n + 1) = 2n$, $f(3n + 2) = 4n + 1$ has a cycle of period 94, with minimum 140, and 94 is not the denominator of any convergent of $\log_2 3$ [3]. We have checked it: in that cycle $K/L = 149/94$, with $|\log_2 3 - 149/94| \approx 1.4 \cdot 10^{-4}$, larger than $1/(2 \cdot 94^2)$, and the condition analogous to (9) does not hold because the minimum is small compared with the length. The example is therefore consistent with Theorem 8.17 and shows that the observed correspondence is not automatic.

Remark 8.21. According to the same bibliography [3], Atkin described a method that determines in principle all cycles of a given period, and checked by computer that, apart from the cycles of periods $1, 2, 5$ and 12, there are no cycles of period less than 200. Corollary 8.18 deals with periods up to 5000; in particular, there is no cycle of period 41, which would correspond to the convergent $65/41$. Simons [11] classifies the cycles of this permutation by their number m of local minima and studies them with linear forms in logarithms: for $m \le 2$ there are no cycles other than (1), $(2, 3)$ and $(4, 5, 7, 9, 6)$; for $m = 3, 4, 5$ he excludes cycles with elements smaller than 10^6 and leaves open, for larger elements, among others, the cases with period and number of terms divisible by 3 equal to $(665, 276)$, $(1330, 552)$, $(1636, 679)$ and $(1995, 828)$. These four cases are among those excluded by Corollary 8.18. The bibliography gives 144 as the minimum of the cycle of period 12; this is a misprint for 44, already pointed out in the OEIS, sequences A185589 and A185590 [4]: the orbit of 144 begins $144, 96, 64, 85, 113, \dots$, and by Proposition 7.1 the number 144 does not belong to any cycle of length at most 5000. According to the same bibliography, Keller [19] proved in 1999 that, for each L, certain periodically linear permutations, one of which is conjugate to g, have only finitely many cycles of period L; Corollary 8.10 obtains this finiteness for g with a different proof. Keller also conjectured that, for large periods, the periods of the cycles are denominators of convergents.

Remark 8.22 (scope of the verification). In the $3x + 1$ problem, searches for cycles rest on the verification that every integer up to a very large bound reaches 1: an integer that reaches 1

cannot be the minimum of a new cycle, and this verification, combined with bounds of the type (7), excludes cycles of very large length [3]. For g there is no analogue. The minimum of a bi-infinite orbit, such as 8 or 14, never falls below itself again, so no finite computation can certify that it is not the minimum of a cycle of arbitrarily large period; this is precisely Collatz's question about 8. In the search of Corollary 8.18, the iteration from a minimum of this kind never stops by falling below it, and the cost of the verification grows with the period without the bound being extensible in a way comparable to the $3x + 1$ problem. Tavares [17] makes an observation of the same kind: unlike the $3x + 1$ problem, where computers verify the conjecture for a huge number of values, it is not clear how a computer could help to choose an answer to the question of whether g has only finitely many orbits. The interest of Corollary 8.18 lies in the particular cases it excludes, among them those left open by Simons, rather than in the size of the bound.

9 The open problem

Problem 9.1 (Collatz, 1932)**.**

1. Does 8 belong to a finite cycle of g? Equivalently, by Proposition 4.1, is the forward orbit $\{g^k(8) : k \geq 0\}$ bounded?
2. Does g have cycles other than the four of Proposition 5.1?

What is proved in this note is that g is a permutation with explicit inverse, that its orbits obey the dichotomy of Proposition 4.1 and that the four cycles listed exist, together with the characterization of cycles by their words of residues (Theorem 8.8) and the constraints of Section 8 on arbitrary cycles. The data of Observation 6.1 and Proposition 7.1 are computational verifications. Corollary 8.18, according to which the only cycles of period at most 5000 are the four known ones, is a computer-assisted proof: the constraints of Section 8 reduce the problem to an explicit finite set of candidates, which is checked exhaustively with exact integer arithmetic. Remark 6.2, which points to the orbit of 8 being infinite, is heuristic. The correspondence of the cycles of lengths 5 and 12 with convergents of $\log_2 3$ (Observation 8.20) is, for now, an observed fact, already noted by Shanks in 1965. Both questions remain open.

A Code

```
def g(n):
    q, r = divmod(n, 3)
    return (2*q, 4*q + 1, 4*q + 3)[r]

def ginv(m):
    if m % 2 == 0:
        return 3*m // 2
    return (3*m + 1)//4 if m % 4 == 1 else (3*m - 1)//4

# comprobación de la inversa
assert all(ginv(g(n)) == n and g(ginv(n)) == n for n in range(1, 10**6))
```

```
# búsqueda de ciclos: mínimo m <= 10^5, longitud <= 5000
ciclos = []
for m in range(1, 10**5 + 1):
    x, k = g(m), 1
    while x > m and k < 5000:
        x, k = g(x), k + 1
    if x == m:
        ciclos.append((m, k))
print(ciclos)   # [(1, 1), (2, 2), (4, 5), (44, 12)]

# órbita de 8: 10^6 pasos en cada sentido
import sys; sys.set_int_max_str_digits(0)
x = y = 8
for _ in range(10**6):
    x, y = g(x), ginv(y)
print(len(str(x)), len(str(y)))   # 24282 25655

# Corolario: tamiz exacto (enteros) para L <= 5000, m > 10^5
def c1(L, a, m): return 2**a * (4*m - 1)**(L - a) <= 3**L * m**(L - a)
def c2(L, a, m): return 3**L * m**(L - a) <= 2**a * (4*m + 1)**(L - a)
def admisibles(L, m):      # c1 vale si a >= a1; c2 si a <= a2
    lo, hi = 0, L + 1
    while lo < hi:
        mid = (lo + hi) // 2
        lo, hi = (lo, mid) if c1(L, mid, m) else (mid + 1, hi)
    a1, lo, hi = lo, -1, L
    while lo < hi:
        mid = (lo + hi + 1) // 2
        lo, hi = (mid, hi) if c2(L, mid, m) else (lo, mid - 1)
    return list(range(a1, lo + 1))
m0 = 100001
def cota(L, a):            # mayor m que cumple c1 y c2
    lo, hi = m0, 10**9
    while hi - lo > 1:
        mid = (lo + hi) // 2
        ok = c1(L, a, mid) and c2(L, a, mid)
        lo, hi = (mid, hi) if ok else (lo, mid)
    return lo
pares = [(L, a, cota(L, a)) for L in range(1, 5001)
         for a in admisibles(L, m0)]
print(len(pares), sum(L <= 1000 for L, a, mm in pares))
# 53 2  -> 2 pares con L <= 1000 y 51 con 1001 <= L <= 5000
print(pares[:2])  # [(665, 276, 2227739), (971, 403, 145107)]
cotas = [mm for L, a, mm in pares if L > 1000]
print(min(cotas), max(cotas))   # 100593 2227739

def es_minimo_de_ciclo(m, L, a):  # con poda por composición
    x, k, z = m, 0, 0
    while True:
        if x % 3 == 0:
            z += 1
```

```
            if z > a: return False
        x, k = g(x), k + 1
        if x <= m: return x == m
        if k - z > L - a or k >= L: return False
for L, a, mmax in pares:          # unos 40 minutos en total
    assert not any(m % 6 in (2, 4) and es_minimo_de_ciclo(m, L, a)
                   for m in range(m0, mmax + 1))
print('ningún ciclo nuevo de periodo <= 5000')
```

Note on the use of artificial intelligence tools

Artificial intelligence tools were used in the preparation of this note as assistance in the drafting, structuring and presentation of the text. The authorship, the mathematical content and the conceptual direction of the work are the author's, who has checked all of the content and is responsible for it; the artificial intelligence tools acted as editorial and formalization support, not as a source of the ideas or the results. The computations on which Proposition 7.1, Observation 6.1 and Corollary 8.18 rest can be reproduced with the code in Appendix A.

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